An Error Criterion for the Pointing of Axially Symmetric Spacecraft Payloads

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A scalar function of the attitude error covariance matrix is derived for use as a criterion in the pointing of axially symmetric spacecraft payloads such as antennas and telescopes. Rotation errors about the payload boresight are not penalized. Mathematically, the performance criterion proposed is shown to equal the sum of the two largest eigenvalues of the attitude error covariance matrix. The physical significance of this cost function and its relationship to more conventional error criteria are described. Several applications are presented. It is concluded that the cost function selected should depend upon the application since, in general, the optimum solution varies with different cost functions.

Introduction

IT is often necessary to determine accurately the orientation of a spacecraft antenna or sensor and point it in a given direction or set of directions. In such a case (Fig. 1), one seeks to minimize the pointing error in the directions of interest; the sensor, for example, being fixed in one coordinate frame and its attitude referred to a second reference frame. This second coordinate frame is typically a reference system whose orientation in space is determined by other means, such as stellar sightings. Even in specific situations, most investigators choose to structure their analysis around one of three conventional measures of the pointing error covariance matrix P_ϵ. It is the writer's opinion, however, that an error criterion which is "tailor-made" to the pointing application is more meaningful.

Since many payloads and experiments, such as antennas and telescopes, exhibit axial symmetry about a boresight axis, a cost function is derived here which is particularly suited to such applications, in that rotation errors about the boresight axis are not penalized. Based on a reasonably thorough review of the literature, the error criterion of this paper is believed to be novel as well as practically useful.

The development of an error criterion for axially symmetric payloads proceeds from a geometric viewpoint. The effect of an attitude error on a given pointing vector can be broken into two complementary components: One component is a rotation error about an axis orthogonal to the pointing vector $\hat{v}$, resulting in misdirection of this vector, and the other is a rotation error about the nominal pointing vector axis, which is inconsequential in many applications, such as the pointing of an axially symmetric antenna. It can then be demonstrated that this decomposition of the pointing error into two orthogonal components has a statistical equivalent, which forms the basis for the cost functions considered in this paper. Following the development of the error criterion for axially symmetric payloads, its significance and relationship to other possible cost functions are discussed.

The Problem Setting

It is assumed that the principal parameters of concern are the small angle rotation errors $\epsilon^T = (\epsilon_1, \epsilon_2, \epsilon_3)$ describing the misalignment between the two coordinate frames of interest. The error e_ϵ in estimating ϵ is assumed to be zero mean and, thus, statistically typified by its covariance matrix

$$P_\epsilon \triangleq E\{e_\epsilon e_\epsilon^T\} \tag{1}$$

It is generally desirable to minimize some measure of the matrix P_ϵ. Conventional scalar measures are

$$J_T \triangleq Tr\{P_\epsilon\} = \sum_{i=1}^{3} \lambda_i$$

$$J_\Delta \triangleq Det\{P_\epsilon\} = \prod_{i=1}^{3} \lambda_i$$

and

$$J_{\max} \triangleq \lambda_{\max}\{P_\epsilon\}$$

where λ_i is an eigenvalue of P_ϵ with Tr and Det being the trace and determinant, respectively. These and other measures are considered in Refs. 1-12.

On the other hand, it may be desirable to minimize some particular function $g(\epsilon)$ of the alignment errors. If the function is linear,

$$g(\epsilon) = a^T \epsilon \tag{2}$$

where

$$a^T = (a_1, a_2, a_3) \tag{3}$$

the error in $g(\epsilon)$ will be given by

$$e_g = a^T e_\epsilon \tag{4}$$

The covariance (a scalar) of e_g is

$$p_g = E\{e_g e_g^T\} = a^T P_\epsilon a \tag{5}$$

In such a case, P_ϵ will still be required, but it will be desired to minimize the scalar p_g. If $g(\epsilon)$ is a nonlinear function, since e_ϵ is a small quantity, the formulation remains essentially the same, with the linearized sensitivity coefficient vector $[\partial g(\epsilon)/\partial \epsilon]_{\epsilon=0}$ replacing a in Eq. (4).

The Deterministic Case

Figure 2 depicts the pointing error of a unit vector $\hat{v}$ fixed in a coordinate system S which has been misaligned. This misalignment may be represented as an error vector $\epsilon^T = [\epsilon_1 \epsilon_2, \epsilon_3]$. For simplicity in what follows, no further distinction will be made between an attitude error ϵ and an error e_ϵ in the knowledge of such an error. Since its com-

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ponents are assumed to be infinitesimal (actually small) rotations, the misdirection $\boldsymbol{\delta}$ of the vector $\hat{v}$ can be represented by a small rotation ϕ of magnitude $\|\epsilon \times \hat{v}\|$ about a unit vector $\hat{n}$ directed along the vector $\hat{v} \times (\epsilon \times \hat{v})$, as shown in Fig. 2. Obviously, $\hat{n}$ is orthogonal to both $\hat{v}$ and $\boldsymbol{\delta}$.

In vector-matrix notation, $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta} = [I + \tilde{E}]\hat{v} - \hat{v} = \tilde{E}\hat{v} \tag{6}$$

where in terms of the components ϵ_i of ϵ

$$\tilde{E} \triangleq \begin{bmatrix} 0 & \epsilon_3 & -\epsilon_2 \\ -\epsilon_3 & 0 & \epsilon_1 \\ \epsilon_2 & -\epsilon_1 & 0 \end{bmatrix} \tag{7}$$

The magnitude of ϕ is given by

$$|\phi| = \tan^{-1}\left\{\frac{\|\delta\|}{\|\hat{v}\|}\right\} \cong \|\delta\| = (\delta^T\delta)^{1/2} \tag{8}$$

where the notation $\|\boldsymbol{x}\|$ denotes the magnitude of the vector $\boldsymbol{x}$. Thus, from Eqs. (6) and (8),

$$\phi^2 = \delta^T\delta = \hat{v}^T\tilde{E}^T\tilde{E}\hat{v} = \hat{v}^T[\epsilon^T\epsilon I - \epsilon\epsilon^T]\hat{v} \tag{9}$$

The angular rotation error θ induced by ϵ about the vector $\hat{v}$ is just the projection of ϵ onto $\hat{v}$ given by $\theta = \epsilon^T\hat{v}$, so that

$$\theta^2 = (\epsilon^T\hat{v})^2 = (\hat{v}^T\epsilon)(\epsilon^T\hat{v}) \tag{10}$$

Adding Eqs. (9) and (10) gives

$$\theta^2 + \phi^2 = \hat{v}^T\epsilon^T\epsilon\hat{v} = (\epsilon^T\epsilon)(\hat{v}^T\hat{v}) = \epsilon^T\epsilon = \|\epsilon\|^2 \tag{11}$$

which is just the square of the magnitude of pointing error vector ϵ.

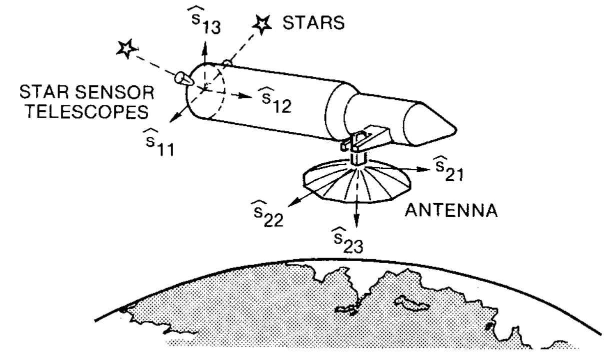

Fig. 1 Spacecraft with typical axially symmetric payload.

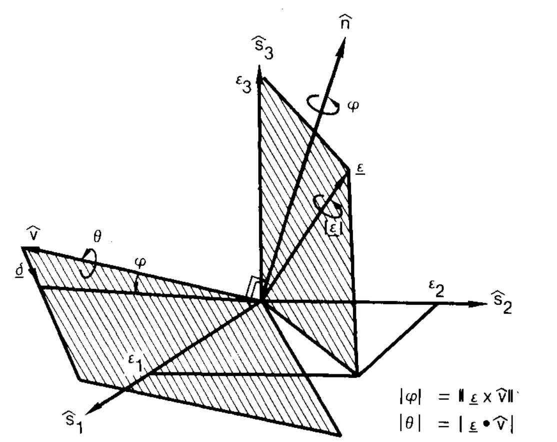

Fig. 2 Pointing error due to coordinate system misalignment.

Since θ is the projection of ϵ along $\hat{v}$ and ϕ is a rotation about $\hat{n}$ which is normal to $\hat{v}$ and, furthermore, the magnitude of ϵ is related to θ and ϕ via Eq. (11), it follows that ϵ can be considered to have two orthogonal components $\theta\hat{v}$ and $\phi\hat{n}$.

The fact that $|\phi| = \|\epsilon \times \hat{v}\|$ follows from Eqs. (6) and (8). In addition, $\theta = \epsilon \cdot \hat{v}$. Consequently, in the deterministic case, the misdirection ϕ of the line-of-sight vector $\hat{v}$ is maximized when $\hat{v}$ lies anywhere in the plane normal to ϵ, in which case the twist error about $\hat{v}$ is zero. Alternatively, the twist θ is maximized when $\hat{v}$ is colinear with ϵ, resulting in a zero misdirection error ϕ.

The Statistical Case

In reality, ϵ is not a known quantity, but its statistical description can often be determined. Estimates of ϵ will normally have a zero mean, that is, $E\{\epsilon\} = \mathbf{0}$. Thus the expectation of θ, for example, is zero.

However, in a practical sense, a positive rotation error in one instance does not cancel an equal magnitude negative rotation error in another. Consequently, it is the magnitude $|\theta|$ of θ which is of interest when rotational errors are a concern. The mean value of $|\theta|$ is not zero; nor is the mean value of $|\phi|$. The quantitites $\theta = \hat{v}^T\epsilon$ and $\phi\hat{n} = \hat{v} \times \epsilon$ both represent linear operations on the vector ϵ. Thus, if the components of ϵ are zero mean multivariate Gaussian variables, then so are θ and the components of $\hat{v} \times \epsilon$. This fact could be used to compute the mean values of $|\theta|$ and $|\phi|$ for use as measures of these errors in such a case.

However, more meaningful and readily computed measures of $|\theta|$ and $|\phi|$ are their mean-square values. These are obtained by taking the expectations of Eqs. (9), (10), and (11), respectively, to obtain the scalar measures

$$\begin{aligned} p_\theta &\triangleq E\{|\theta|^2\} = \hat{v}^T P_\epsilon \hat{v} \\ p_\phi &\triangleq E\{|\phi|^2\} = \hat{v}^T[T_r\{P_\epsilon\}I - P_\epsilon]\hat{v} \\ p_\epsilon &\triangleq E\{\|\epsilon\|^2\} = Tr\{P_\epsilon\} = p_\theta + p_\phi \end{aligned} \tag{12}$$

where use has been made of the definition of P_ϵ in Eq. (1). Thus, depending on the application, p_θ,p_ϕ, or their sum may be taken as the measure of the pointing error of concern. It is noted that whereas p_θ and p_ϕ depend upon the pointing direction $\hat{v}$, p_ϵ does not.

The Cost Function J_ϕ

In general, we may not know the pointing direction, or it may be necessary to point over a variety of directions. In such a case, it is desirable to have a cost function that is independent of $\hat{v}$. Several such cost functions are developed below, and others are possible; for example, see Kalman et al.[6]

Meaningful measures of $|\theta|$ and $|\phi|$ are the maximum values of their respective mean-square values taken over all possible pointing directions $\hat{v}$. It is well known that when M is a real symmetric matrix,

$$\max_{\hat{v}}\{\hat{v}^T M \hat{v}\} = \max_{v \neq 0}\{v^T M v / v^T v\} = \lambda_{\max}\{M\} \tag{13}$$

and that the $\hat{v}$ which yields this maximum is the unit eigenvector of M corresponding to its maximum eigenvalue $\lambda_{\max}\{M\}$. Thus, applying Eq. (13) to Eq. (12) gives

$$\begin{aligned} J_\theta &\triangleq \max_{\hat{v}}\{p_\theta\} = \lambda_{\max}\{P_\epsilon\} \\ J_\phi &\triangleq \max_{\hat{v}}\{p_\phi\} = \lambda_{\max}\{Tr\{P_\epsilon\}I - P_\epsilon\} \\ J_\epsilon &\triangleq \max_{\hat{v}}\{p_\epsilon\} = Tr\{p_\epsilon\} \end{aligned} \tag{14}$$

It is noted that both J_θ and J_ϵ are among the three common error measures given earlier; however, J_ϕ is not.

Since J_ϕ is the measure of pointing error which is independent of rotation errors about the payload boresight axis, it is desirable to express it in a simpler, more useful form. The eigenvalues of the matrix $[Tr\{P_\epsilon\}I-P_\epsilon]$ are defined by

$$\det\{Tr\{P_\epsilon\}I-P_\epsilon-\lambda' I\}=\det\{P_\epsilon-(Tr\{P_\epsilon\}-\lambda')I\}=0 \quad (15)$$

Defining

$$\lambda=Tr\{P_\epsilon\}-\lambda' \quad (16)$$

it is seen that λ is an eigenvalue of P_ϵ. Thus, from Eqs. (15) and (16),

$$\lambda'\{Tr\{P_\epsilon\}I-P_e\}=Tr\{P_\epsilon\}-\lambda\{P_\epsilon\} \quad (17)$$

and it follows that

$$J_\phi=Tr\{P_\epsilon\}-\lambda_{\min}\{P_\epsilon\}=\lambda_3\{P_\epsilon\}+\lambda_2\{P_\epsilon\} \quad (18)$$

where $\lambda_3\{P_\epsilon\}\geq\lambda_2\{P_\epsilon\}\geq\lambda_1\{P_\epsilon\}$.

From the form of Eq. (16), it is also obvious that the eigenvectors of $[Tr\{P_\epsilon\}I-P_\epsilon]$ are the same as those of P_ϵ, where the eigenvector corresponding to $\lambda_{\max}$ is the same as that corresponding to $\lambda'_{\min}$ and vice versa. If $\hat{e}_1$ is the unit eigenvector of P_ϵ corresponding to λ_i, then it follows that $\hat{v}=\hat{e}_3$ maximizes p_θ, whereas $\hat{v}=e_1$ maximizes p_ϕ. As previously mentioned, p_ϵ is the same for all $\hat{v}$ directions.

The covariance ellipsoid is defined here by

$$v^T P_\epsilon v=1 \quad (19)$$

although the alternative definition $v^T P_\epsilon^{-1} v=1$ is often found. Since P_ϵ is a real symmetric matrix, it is well known that it may be diagonalized by the similarity transformation $v=\Phi y$, where Φ is an orthonormal rotation matrix (see page 298 of Ref. 13). Substituting this value into Eq. (19) gives

$$y^T\Phi^T P_\epsilon \Phi y=y^T\Lambda_\epsilon y=1 \quad (20)$$

where Λ_ϵ is a diagonal matrix containing the eigenvalues of P_ϵ. The covariance ellipsoid corresponding to P_ϵ is shown in Fig. 3.

It should be noted that the major axis of the ellipsoid corresponds to the minimum eigenvalue λ_1 of P_ϵ, and that the minor axis corresponds to λ_3, the largest eigenvalue of P_ϵ. Since the similarity transformation leaves the trace of the transformed matrix unchanged, and since the diagonal components of P_ϵ, p_{ii} are given by

$$p_{ii}=\sigma^2_{\epsilon_i} \qquad i=1,2,3 \quad (21)$$

it follows that

$$Tr\{P_\epsilon\}=\sigma^2_{\epsilon_1}+\sigma^2_{\epsilon_2}+\sigma^2_{\epsilon_3}=Tr\{\Lambda_\epsilon\}=\lambda_1+\lambda_2+\lambda_3 \quad (22)$$

although the correspondence in Eq. (22) does not hold on a term-by-term basis. As a consequence of Eq. (22),

$$J_\epsilon=\lambda_1\{P_\epsilon\}+\lambda_2\{P_\epsilon\}+\lambda_3\{P_\epsilon\} \quad (23)$$

In summary, the cost functions J_θ, J_ϕ, and J_ϵ are given by

$$J_\theta=\lambda_3\{P_\epsilon\}$$

$$J_\phi=\lambda_3\{P_\epsilon\}+\lambda_2\{P_\epsilon\} \qquad \lambda_3\geq\lambda_2\geq\lambda_1 \quad (24)$$

$$J_\epsilon=\lambda_3\{P_\epsilon\}+\lambda_2\{P_\epsilon\}+\lambda_1\{P_\epsilon\}$$

Each of these cost functions corresponds to a measure of the effect of attitude errors on the accuracy of properly orienting a satellite payload. If only the orientation about the boresight axis is of interest, J_θ should be used. In the more typical case where rotation errors about the boresight axis are of no consequence but the orientation of this axis is critical, J_ϕ is of interest. If both are important, J_ϵ, which measures both effects, is the cost function to be employed. In the case of J_ϕ, the mean-square error to be minimized is the maximum possible in the sense that it corresponds to that pointing direction which will experience the maximum mean-square deflection for a statistically defined set of attitude errors. Similarly, J_θ corresponds to that pointing direction for which the mean-square rotation error about the boresight axis is a maximum.

Examples

Two examples have been selected which serve to demonstrate the significance which the selection of an error criterion may have in practice. It will be seen that in some instances the choice can be inconsequential, but in others it may be crucial.

The effect of the cost function employed depends upon the structure of the particular attitude error covariance matrix involved and thus on the problem being studied. In order to conserve space, both examples treated here are of the same generic form arising from a particular class of attitude error optimization problems. The principle differences between the examples are due to the constraints under which the optimization is performed.

The Dynamic Alignment Monitor Problem

The class of problems considered is related to a technique for on-orbit angular alignment of two spacecraft coordinate systems such as those of Fig. 4. The technique is called "dynamically activated alignment" by the writer, and it is sometimes known as rate matching. In this alignment scheme, it is assumed that a set of three single-axis rate gyros is located at each of the coordinate systems of interest, with one gyro in each set having its sensitive axis along one of the three orthogonal references axes. To make the examples considered analytically tractable, a number of assumptions are made here. The interested reader can refer to Ref. 14 for a more complete treatment of the problem.

The first assumption is that the gyros are idealized instruments having no scale factor or bias errors. Another assumption is that the gyros are mounted on a rigid body which has no significant flexible-body modes. The two coordinate systems are taken to be nominally aligned so that the direction cosine matrix relating their orientations is given by

$$T=I+\tilde{E} \quad (25)$$

where $\tilde{E}$, as defined in Eq. (7), represents the infinitesimal misalignment of the two systems. It is desired to determine the components ϵ_i of $\tilde{E}$.

In order to measure this misalignment, the spacecraft is set into angular motion, and the body-fixed components of the

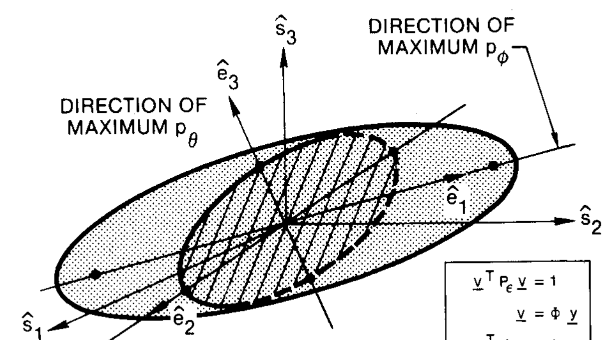

Fig. 3 P_ϵ covariance ellipsoid showing directions of maximum p_θ and p_ϕ.

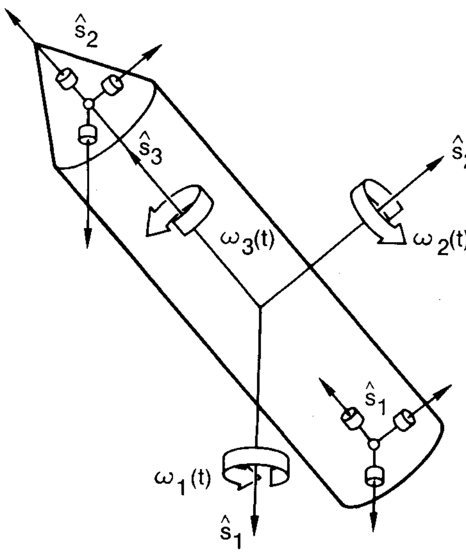

Fig. 4 Dynamic alignment monitor example.

angular velocity vector

$$\omega^T(t) \triangleq [\omega_1(t) \quad \omega_2(t) \quad \omega_3(t)] \tag{26}$$

as measured by the two sets of gyros are compared on a component-by-component basis. If there is no misalignment, corresponding gyros in each of the two frames will give identical readings except for noise. If the motion of the spacecraft is properly chosen, differences in those measurements can be used to solve for the misalignments ϵ_i about each of the three coordinate axes.

If product terms in the misalignment error and gyro measurement noises are neglected as second-order effects and the gyros are assumed to have noises which are zero mean, statistically independent, and of unit variance, it can be shown that the covariance matrix P_ϵ of the residual misalignment errors following a spacecraft maneuver is given by

$$P_\epsilon^{-1} = \begin{bmatrix} c_{33}+c_{22} & -c_{12} & -c_{13} \\ -c_{12} & c_{33}+c_{11} & -c_{23} \\ -c_{13} & -c_{23} & c_{11}+c_{22} \end{bmatrix} \tag{27}$$

where

$$c_{ij} \triangleq \int_0^T \omega_i(t)\omega_j(t)\,\mathrm{d}t \tag{28}$$

Although not readily apparent, it is shown in Refs. 14 and 15 that, because the alignment measurements are weighted according to the inverses of the gyro noise variances (here assumed to have unity magnitude), the units of P are square radians.

In order to reduce the complexity of the examples which follow, the time profiles of the angular velocity components $\omega_i(t)$ are taken to be orthogonal over the measurement interval so that $c_{ij}=0$ if $i\neq j$. This diagonalizes the matrix of Eq. (27) so that its inverse is easily computed as

$$P_\epsilon = \begin{bmatrix} \dfrac{1}{c_2+c_3} & 0 & 0 \\ 0 & \dfrac{1}{c_3+c_1} & 0 \\ 0 & 0 & \dfrac{1}{c_1+c_2} \end{bmatrix} \tag{29}$$

where, for simplicity of notation, c_i replaces c_{ii} in the remainder of this paper.

The examples which follow explore the nature of P_ϵ for various sets of problem constraints on the c_i. For P_ϵ as given by Eq. (29), the various cost functions are

$$J_\Delta = \frac{1}{(c_2+c_3)(c_3+c_1)(c_1+c_2)}$$

$$J_\epsilon = \frac{1}{c_2+c_3} + \frac{1}{c_3+c_1} + \frac{1}{c_1+c_2}$$

$$J_\theta = \frac{1}{c_i+c_j} \qquad \text{for} \quad c_i \le c_j \le c_k$$

$$J_\phi = \frac{1}{c_i+c_j} + \frac{1}{c_i+c_k} \qquad \text{for} \quad c_i \le c_j \le c_k \tag{30}$$

Independent Three-Axis Control

As an example of applying the various error criteria to the minimization of residual errors following a dynamic alignment maneuver, consider the case of a spacecraft with independent three-axis control where each of the three angular velocity components obeys an amplitude constraint of the form $|\omega_i(t)| \le \Omega_i$. As a result,

$$c_i \le (\Omega_i)^2 T \triangleq s_i \tag{31}$$

where T is the duration of the maneuver interval.

Obviously, J_Δ, J_ϵ, and J_ϕ are minimized when the c_i assume their maximum possible values so that the corresponding optimum values J_Δ^0 and J_ϵ^0 are given by Eq. (30), where the optimizing values $c_i^0 = s_i$; $i=1,2,3$. Assuming that $s_1 \le s_2 \le s_3$, further inspection of Eq. (30) reveals that J_θ^0 corresponds to $c_i^0 = s_1$, $c_j^0 = s_2$, and $s_2 \le c_k^0 \le s_3$.

As expected, the optimum values of the cost functions differ because each function corresponds to a different function of the eigenvalues of P_ϵ. What is interesting is that J_Δ, J_ϵ and J_ϕ are all optimized for the same values of the c_i, whereas for J_θ, the corresponding values of c_i and c_j are identical, but c_k may assume any value in the interval between s_2 and s_3 without changing the value of J_θ^0.

Coupled Three-Axis Control

The next example is somewhat more complex. Consider a spacecraft under the influence of momentum vector control. The alignment errors are taken to be defined about the three principal inertia axes of the spacecraft, which are assumed to coincide with the orientation of the alignment coordinate systems S_1 and S_2 of Fig. 4. In the general case, the dynamics of the spacecraft are constrained by the relation

$$h(t) = I\omega(t) \tag{32}$$

where h is the control momentum vector and I is the (diagonal) inertia matrix having diagonal elements I_1, I_2, and I_3. In order to simplify the example, it is also assumed that the control moment gyro (CMG) mass is a sphere centered at the spacecraft's center of gravity. For this example, the magnitude of h given by

$$h^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \tag{33}$$

is assumed to be a fixed constant, but the momentum vector may be continuously oriented so as to alter the values of the $\omega_i(t)$; $i=1,2,3$. Integrating Eq. (33) over the interval $t \in [0,T]$ gives the constraint relation on the c_i. This may be written in normalized form as

$$a_1c_1 + a_2c_2 + a_3c_3 = 1 \tag{34}$$

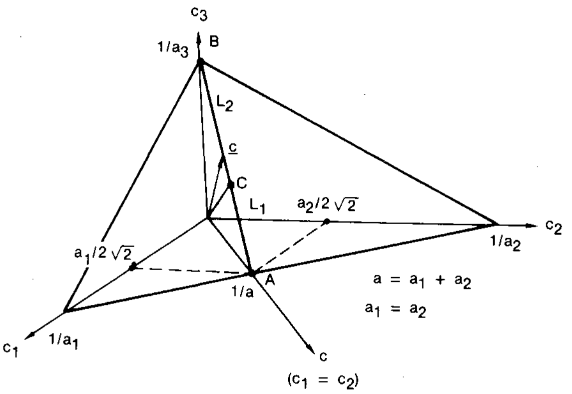

Fig. 5 Constraint relation for coupled three-axis control.

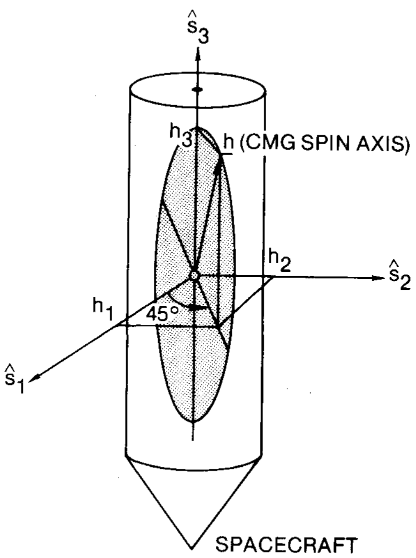

Fig. 6 Spacecraft having two equal angular velocity components.

where $a_i \triangleq I_i^2/(h^2T)$. This constraint is portrayed in Fig. 5. The tip of the c vector, $c^T = (c_1 c_2 c_3)$, must lie on the surface of the plane shown in Fig. 5. Note that, by definition of the $c_i \triangleq c_{ii}$ in Eq. (28), $c_i \geq 0$; $i = 1.2.3$.

This three-dimensional problem of minimizing the cost functions of Eq. (30) is analytically solvable; however, since the problem is of some importance and the space required just to summarize the approach and the results is prohibitive, it is planned to make this problem the subject of a separate paper for future consideration. To simplify matters somewhat, c_2 will be taken equal to c_1, thereby reducing the problem to two dimensions in c-space. The resulting problem has several physical realizations, one of which is an axially symmetric spacecraft with a single-gimbal, constant-speed CMG having its spin axis (h axis) oriented as shown in Fig. 6.

In this case, the CMG is mounted so that its spin axis lies in a plane which is oriented at 45 deg with respect to the $\hat{s}_1$ and $\hat{s}_2$ axes of the alignment reference frames. Thus $h_2(t) = h_1(t)$ at any instant of time so that for equal transverse inertia components $I_2 = I_1$, it follows that $\omega_2(t) = \omega_1(t)$ and, consequently, $c_2 = c_1$. (Obviously, it is possible to have $c_1 = c_2$ without requiring that $\omega_1(t) = \omega_2(t)$, since only their time integrals must be equal.) As a result, Eq. (34) may be written as

$$c_3 = (1 - ac)/a_3 \tag{35}$$

where $a \triangleq a_1 + a_2 = 2a_1$. This constraint is shown in Fig. 5 as the line AB, resulting from the intersection of the plane $c_1 = c_2$ within the three-dimensional constraint plane of Eq. (34).

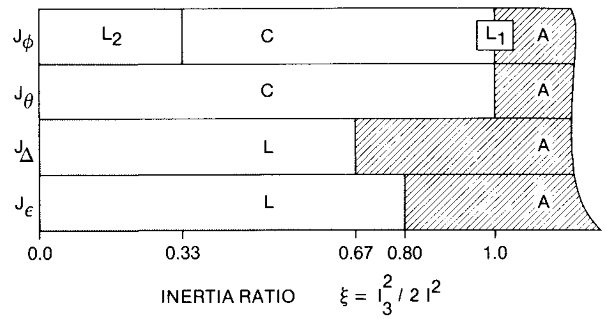

Fig. 7 Location of cost function minima as a function of ξ.

It should be noted that the constraint $c_i \geq 0$ is depicted in Fig. 5 by points A and B. Point C corresponds to $c_3 = c$ and divides the constraint space into two segments L_1 and L_2. In the case of J_ϕ and J_ϵ, the cost function is not only nonlinear, but it is also defined differently on each of the segments. These open line segments are defined by

$$L_1: \frac{1}{a + a_3} < c < \frac{1}{a}$$

$$L_2: 0 < c < \frac{1}{a + a_3} \tag{36}$$

The approach employed in determining the cost function minima is analogous for all four error criteria of Eq. (30), differing only in detail. First, one determines the local minima, if they exist, on the line segments of interest in the constrained (c_3, c) space. In doing so, the constraint of Eq. (35) is eliminated from separate consideration by substituting for c_3 into the cost function of interest. Any resulting local minima must then be compared with the values J_A, J_B, and J_C of the cost function at the points A, B, and C, respectively, to determine the global minimum. Details are given in Ref. 15.

The results of such an analysis for the four cost functions are summarized in Table 1 as a function of the inertia factor ξ defined by

$$\xi \triangleq \frac{a_3}{a} = \frac{I_3^2}{I_1^2 + I_2^2} = \frac{I_3^2}{2I^2} \tag{37}$$

It is noted that the value of J_B is infinity for all four functions and is consequently excluded from further consideration. The location $c°$ of any local minimum which exists on the constraint line segments L_1 and L_2 of Fig. 5 is given in the table. Also given is the associated value $J°$ of the cost function at the local minimum. In the case of J_θ and J_ϕ, when a local minimum exists, it always occurs on the line segment L_2 since both functions are monotonic functions of c on line segment L_1. (A minor exception exists for $\xi = 1$, where J_ϕ is constant over L.)

Having determined the local minima of each of the cost functions J_ϕ, J_θ, J_Δ, and J_ϵ and the values of these cost functions at points A and B as well as at the local minima, the locations of the global minima of each of these error criteria are compared with the aid of Figs. 7 and 8. Figure 7 indicates, in bar-chart form, the location of the global minimum for each of these error criteria as a function ξ, which is one-half the square of the spacecraft axial-to-transverse inertia ratio. The notations C and A refer to the endpoints of the admissable region in c-space, as shown in Fig. 5. L_1 L_2 and L refer to linear regions also shown in Fig. 5, L being the union of L_1 and L_2. The inset box for J_ϕ containing L_1 is meant to indicate that for $\xi = 1$, the minimum of J_ϕ lies anywhere along the line L_1. In the case of the other line segments L and L_2, the minimum lies somewhere along the segment in question. It is important to note that for $\xi > 1$, the location of the minimum cost, regardless of the criterion used, is at point A.

Table 1 Cost function values J_A and J_C at boundary points and local minima J^0 at locations c°

Cost function	J_A	J_C	J°	c°
J_ϕ	$2a$	$(1+\xi)a$	$\left[\sqrt{\xi}+\frac{\sqrt{1-\xi}}{2}\right]^2 a$	$\frac{1}{a}\left[\frac{1}{(1-\xi)+\sqrt{2\xi(1-\xi)}}\right]$
J_θ	a	$\frac{(1+\xi)a}{2}$	N/A	N/A
J_ϵ	$\frac{5a}{2}$	N/A	$\frac{a}{2}[\sqrt{1-\xi}+2\sqrt{\xi}]^2$	$\frac{1}{a}\left[\frac{1}{(1-\xi)+2\sqrt{\xi(1-\xi)}}\right]$
J_Δ	$\frac{a^3}{2}$	N/A	$\frac{27}{8}\xi(1-\xi)^2a^3$	$\frac{1}{3a(1-\xi)}$

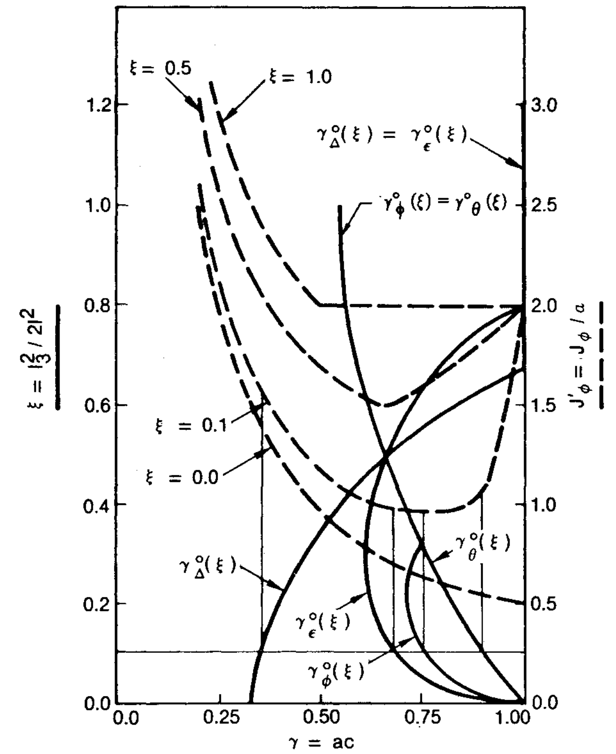

Fig. 8 Sensitivity of J_ϕ to the cost function optimized.

Figure 8 presents two types of information. First, the quantitative locations of the global minima for each of the cost functions, in terms of the normalized distance parameter γ, are represented as a function of the inertia factor ξ by the solid curves. The other four dashed-line curves, one each for $\xi = 0.0$, 0.1, 0.5, and 1.0, give the normalized cost function J'_ϕ as a function of γ.

Thus, if J_ϕ is the error criterion of interest, it is possible from Fig. 8 to determine graphically not only the location and optimum value of J_ϕ for the four given values of ξ but, in addition, the error incurred in J_ϕ if one of the other three error criteria is optimized in lieu of J_ϕ.

As an example, consider the inertia ratio $\xi = 0.1$ as indicated by the corresponding horizontal line in the figure. The intersections of this line with the four γ° curves are extended vertically, as shown, until they intersect the J'_ϕ plot for $\xi = 0.1$. The corresponding ordinate values as read from the right-hand scale indicate the values of J'_ϕ which would result from optimizing each of the four criteria. The error due to using J_Δ is seen to be particularly large.

It is of interest to note that for $\xi = 0.5$ (all three inertias equal), all criteria are optimized for the same value, $\gamma = 2/3$. Also, for $\xi = 1$, J_ϕ and J_θ have minima at $\gamma = 0.5$, whereas the minima of J_Δ and J_ϵ are at $\gamma = 1$. However, in all four cases, the same value of $J'_\phi = 2$ is attained, since J'_ϕ is constant for $0.5 \leq \gamma \leq 1.0$.

The conclusion to be drawn from this example is that one should always employ an error criterion which reflects that quantity of particular interest for the problem at hand. The optimization of a different criterion will, in general, result in a nonoptimal solution as far as the actual criterion of interest, in this case J_ϕ, is concerned.

Summary and Conclusions

An error criterion has been developed which takes special account of the fact that most spacecraft payloads which require pointing, such as antennas, telescopes, lasers, etc., are symmetrical about the boresight axis. This new criterion does not penalize rotation errors about the boresight axis. Moreover, it is a measure of the statistical expectation of the mean-square boresight deflection error for that pointing direction which maximizes this error, given a statistically specified set of spacecraft attitude errors.

The cost function developed in this paper, explicitly for pointing applications, has been compared and related to three more commonly used measures of attitude error, where it was shown that two of these, the maximum eigenvalue and trace of the attitude error covariance matrix, are measures of the rotational orientation errors about the boresight axis and the total angular pointing error, respectively.

Two examples were presented, one for uncoupled and one for coupled three-axis control of a spacecraft. Both examples were based on a situation where dynamically activated alignment of two coordinate systems was sought by employing spacecraft attitude maneuvers which are measured via two sets of three-axis, orthogonally mounted gyro packages.

In the uncoupled case, it was shown that, with one notable exception, the same optimum design of the spacecraft maneuver is obtained, regardless of which of the four error criteria is selected. The case of the coupled three-axis control is quite different. Even for the 2-degrees-of-freedom situation considered, there are qualitative and quantitative differences between the optima achieved as a function of the error criterion employed. It was shown that, in particular, the determinant of the covariance matrix gave an optimum which differed markedly from the other three.

Thus it is recommended that the error criterion selected for use be tailor-made for the application and not arbitrarily selected from any one of a number of covariance matrix functions, all of which are assured to bound the size of the attitude error covariance matrix in some sense. Also, if one is involved with the pointing of an axially symmetric payload where pointing errors about the boresight axis are inconsequential, it is recommended that a cost function equal to the sum of the two largest attitude error covariance matrix eigenvalues be considered.

References

[1]Aoki, M. and Staley, R.M., "On Input Signal Synthesis in Parameter Identification," *Automatica,* Vol. 6, No. 3, May 1970, pp. 431-440.

[2]Diamant, L.S., "The Observability of Unforced Physical Systems by Linear Non-Sequential Estimators in the Validation of Linear Error Analysis," *AIAA Guidance and Control Conference,* Stanford, Calif., Paper 72-876, Aug. 1972, pp. 1-6.

[3]Friedland, B., "On the Calibration Problem," *IEEE Transactions on Automatic Control,* Vol. AC-22, No. 6, Dec. 1977, pp. 899-905.

[4]Hull, K.L. and Stubberud, A.R., "Evaluation of the Performance of a Variance Estimation Algorithm Using Order Statistics," *Third Symposium on Nonlinear Estimation Theory and Its Applications,* San Diego, Calif., Sept. 1972, pp. 103-111.

[5]Johnson, C.D., "Optimization of a Certain Quality of Complete Controllability and Observability for Linear Dynamical Systems," *Journal of Basic Engineering,* Vol. 9, Ser. D, No. 2, June 1969, pp. 228-238.

[6]Kalman, R.E., Ho, Y.C., and Narendra, K.S., "Controllability of Linear Dynamical Systems," *Contributions to Differential Equations,* Vol. 1, No. 2, 1963, pp. 189-213.

[7]Levadi, V.S., "Design of Input Signals for Parameter Estimation," *IEEE Transactions on Automatic Control,* Vol. AC-11, No. 2, 1973, pp. 205-211.

[8]Mehra, R.K., "Optimal Inputs for Linear System Identification," *Thirteenth Joint Automatic Control Conference,* Stanford, Calif., Paper 28-5, Aug. 1972, p. 811-820.

[9]Mehra, R.K., "Optimal Input Signals for Parameter Estimation in Dynamic Systems—Survey and New Results," *IEEE Transactions on Automatic Control,* Vol. AC-19, No. 6, Dec. 1974, pp. 753-768.

[10]Muller, P.C. and Webber, H.I., "Analysis and Optimization of Certain Qualities of Controllability and Observability for Linear Dynamical Systems," *Automatica,* Vol. 8, No. 3, May 1972, pp. 237-246.

[11]Nahi, N.E., and Napjus, G.A., "Design of Optimal Probing Signals for Vector Parameter Estimation," *IEEE Conference on Decision and Control,* Miami, Beach, Fla., Paper W9-5, Dec. 1971, pp. 162-168.

[12]Weidemann, H.L. and Stear, E.B., "Entropy Analysis of Parameter Estimation," *Information and Control,* Vol. 14, No. 6, June 1969, pp. 493-506.

[13]Hohn, F.E., *Elementary Matrix Algebra,* Macmillan, New York, 1967, p. 298.

[14]Coffey, T.C., *Performance Optimization of a Dynamically-Activated Alignment in Bounded Phase-Space,* UCLA Ph.D dissertation, No. 26030, University Microfilms Intl., Ann Arbor, Mich., June 1979.

[15]Coffey, T.C., "An Error Criterion for the Pointing of Axially Symmetric Spacecraft Payloads," Paper 81-173, *AAS/AIAA Astrodynamics Conference,* Lake Tahoe, Nev., Aug. 1981.